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NOISE-INDUCED CHAOS AND PHASE SPACE FLUX: A SAMPLE-THEORETIC STUDY

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Noise-Induced Chaos and Phase Space Flux

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Abstract - We study the effect of additive noise on near-integrable second-order dynamical systems whose unperturbed flows have homoclinic or heteroclinic orbits. The noise is represented by a type of Shinozuka stochastic process capable of arbitrarily closely approximating Gaussian noise with any specified spectrum. We derive a formula for the flux factor applicable for any asymptotic mean stationary excitation. This derivation shows that, to first order, the effect of the external excitation on the system is mediated by a linear filter associated with the system homoclinic or heteroclinic orbit. It also shows that the stationary mean distribution of the filtered excitation determines the average phase space flux. This is true for both random and nonrandom excitations and indicates that, for the dynamical systems considered here, these two classes of excitation play substantially equivalent roles in the promotion of chaos.

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1. Introduction

For some nonlinear multistable dynamical systems operating in the absence of noise, a change in the system parameters can cause the merging of two attractors into a single attractor composed of two subregions, the motion on this attractor being characterized by jumps between the subregions and by a positive largest Lyapounov exponent. Systems exhibiting such deterministic chaotic motion include the Lorenz equations and the Duffing-Holmes oscillator.

Motions with similar irregular jumps can also occur in these systems owing to the presence of noise [1]. In this context the terms "basin-hopping," "noise-induced jumps," and "stochastic chaos" appear in the literature.

Physical and numerical experiments have shown the difficulty of distinguishing between the two types of motion just described [2], [3]. In particular, this difficulty fueled "the controversy on the deterministic or stochastic character of chemical chaos" discussed by Argoul et al. [4, p.83]. Referring to work on the Belousov-Zhabotinskii reaction by Roux et al. [5] and Hudson et al. [6], Argoul et al. argued [4, p.80] that "after measuring the largest Lyapounov exponent, whose positivity confirmed sensitive dependence upon initial conditions, the demonstration of the determinism was complete, despite objections by certain experts in the kinetics of the B.Z. reaction."

In this paper we show that, for certain types of multistable systems and for certain regions of parameter space, such controversies, or conclusions similar to the one just quoted, may be unwarranted. Some systems that, in the absence of noise, have periodic or quasiperiodic behavior, may in fact become sensitive to initial conditions (i.e., exhibit a topological equivalence to the Smale horseshoe map) owing to a change in deterministic forcing, to the introduction of noise, or to both. We show that for a class of systems, the dichotomy between deterministic and stochastic chaos is, in at least one sense, artificial and that these two types of motion may belong to the same class not only phenomenologically but mathematically as well.

Our investigation is restricted to multistable one-degree-of-freedom dissipative systems whose unperturbed counterparts have homoclinic or heteroclinic orbits. For small periodic or quasiperiodic perturbations, the stable and unstable manifolds associated with such orbits separate and may intersect transversely. A necessary condition for tranverse intersections is that the Melnikov function have simple zeros. The system is susceptible to Smale horseshoe-type chaos [7], [8] only in this case. When these intersections occur, they are infinite in number and define tangles with lobes [9], [10]. The turnstile lobes contained therein drive the transport of phase space across the pseudo-separatrix defined by segments of the separated stable and unstable manifolds. The phase space flux is a measure of the amount of phase space transported and, hence, of the system's propensity for chaos - in the words of Beigie et al. [10], a measure of "how chaotic the system is". The larger the flux, the greater is the probability that an orbit originating within a restricted region of phase space (e.g., one corresponding to a potential well) will escape across the pseudo-separatrix bounding that region [10], [11], [12, p.532].

The tangle lobes generally have twisted, convoluted shapes with difficult to determine areas, making analytical calculation of the flux difficult, if not impossible. In the case of small perturbations, however, the lobes are small and roughly convex in shape. For this case, an elegant connection exists between phase space flux and the Melnikov function. Beigie et al.

[10] show that, for small ε, the phase space flux has the asymptotic expansion $\varepsilon\Phi + O(\varepsilon^2)$ where Φ is a certain time-average of the Melnikov function. This connection between phase space flux and the Melnikov function can be exploited to calculate the flux of perturbed, two-dimensional vector fields.

These concepts developed for deterministic perturbations can also be made to apply to systems excited by noise. An interesting attempt to apply Melnikov theory to noise is reported by Bulsara, Schieve, and Jacobs [13]. However, an objection has been raised to their method by the authors [14], [15]. Thus, to specify the Melnikov function for small random perturbations, we turn instead to the method illustrated in [14]. According to this latter approach, the noise process is represented as a harmonic sum with random parameters. This representation defines an ensemble of noise paths. Each noise path (or excitation) belonging to the ensemble is identified by a fixed choice of the random parameters. Since each realization of the noise process - each path - is a harmonic sum, the theory of the generalized Melnikov function and phase space flux is, as already indicated, directly applicable for every fixed choice of noise parameters. A sample theoretic [16, Appendix A] Melnikov treatment of the noise is thus possible. Through such a path-by-path analysis, it is possible to gauge the effect of random excitation on a system and, in particular, on the system's susceptibility to and propensity for Smale horseshoe-type chaos. The purpose of this paper is to present such a study.

There are several well-known noise models which take the form of a harmonic sum, Nyquist noise [17] being perhaps the best-known. Nyquist noise is Gaussian but its paths are not uniformly bounded. Current Melnikov theory is limited to bounded, uniformly continuous (UC) perturbations [18]. Thus, for its application to an ensemble of perturbations, the ensemble must be uniformly bounded. Hence Nyquist noise cannot be used. An alternative noise model studied by Shinozuka [19] [20] does not suffer this drawback. Shinozuka noise is approximately Gaussian (the approximation improves with the number of terms in the sum) with uniformly bounded paths. Moreover, the paths of the Shinozuka noise model are uniformly continuous - across both time and the ensemble. We introduce a variant of the Shinozuka noise model with the additional property that it is ergodic after filtering. With this modification, Shinozuka noise is perfectly suited for Melnikov analysis and is the noise representation we use in this paper.

The paper is organized as follows. In the second section we present our dynamical model and briefly review important features of the Shinozuka noise process with our modification. In the third section we review and discuss the Melnikov function and the phase space flux. The fourth section defines asymptotic mean stationarity for continuous-time functions and processes and describes those of its properties needed in the calculation of the average phase space flux. The fifth section is devoted to the derivation of formulae for the average phase space flux. These formulae are used in the sixth section to study the effect of noise on the propensity of our system model for homoclinic and heteroclinic chaotic behavior. In the seventh section, we comment on the role played by the noise spectrum in these formulae, illustrating these comments with two specific examples, the Duffing-Holmes oscillator and the rf-driven Josephson junction. The last section summarizes our conclusions.

2. Dynamical model and noise representation

Consider the one degree-of-freedom dynamical system governed by the equation of motion

$$\ddot{x} = -V'(x). \tag{1}$$

This system is integrable with Hamiltonian $\dot{x}^2/2 + V(x)$. System (1) is assumed to have a hyperbolic fixed point connected to itself by a homoclinic orbit or two hyperbolic fixed points connected by a heteroclinic orbit. Introducing small damping and external forcing terms into (1), we obtain

$$\ddot{x} = -V'(x) + \varepsilon[\gamma g(t) + \sigma G_t - k\dot{x}] \tag{2}$$

as our dynamical model. Here g and G represent deterministic and stochastic forcing functions, respectively. g is assumed to be bounded, $|g(t)| \leq 1$, and UC. The nonnegative parameters σ, γ, k fix the relative amounts of forcing and damping. Note that our model is restricted to additive external forcing. This is for simplicity of exposition. The model readily generalizes to multiplicative external forcing by considering $\gamma = \gamma(x, \dot{x})$ and $\sigma = \sigma(x, \dot{x})$ to be functions of the phase space coordinates $(x, \dot{x})$. With appropriate conditions imposed on the functions σ and γ, our method of calculation of phase space flux accommodates this generalization without change. We treat the near-integrable case of (2), $\varepsilon \rightarrow 0$.

The Shinozuka representation [19], [20] of noise is

$$\sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos(\nu_n t + \phi_n) \tag{3}$$

where $\{\nu_n, \phi_n; n=1,2,\ldots,N\}$ are independent random variables defined on a probability space $(\Omega, \mathbb{B}, P)$, $\{\nu_n; n=1,2,\ldots,N\}$ are nonnegative with common distribution Ψ_o, $\{\phi_n; n=1,2,\ldots,N\}$ are identically uniformly distributed over the interval $[0, 2\pi]$, and N is a fixed parameter of the model. This noise process has (one-sided) spectral distribution $2\pi\Psi_o$ [19]. Thus the noise model in (3) can be made to have any given spectrum. The origin and an early use of this noise representation are described in [19].

Consider a randomly weighted version G of the noise sum in (3),

$$G_t = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \frac{1}{K(\nu_n)} \cos(\nu_n t + \phi_n). \tag{4}$$

Let $2\pi\Psi$ be the desired spectrum of G, where Ψ is a continuous probability distribution. Assume K in (4) is positive and bounded away from zero, $K(\nu) \geq K_m > 0$ a.e. $d\Psi$. Also assume $\kappa < \infty$ where

$$\kappa \equiv \int_0^\infty K^2(\nu)\, \Psi(d\nu).$$

Real functions K meeting these conditions are said to be Ψ-admissible. Let the distribution Ψ_o of the angular frequencies ν_n in (4) have the form

$$\Psi_o(A) = \frac{1}{\kappa} \int_A K^2(\nu)\, \Psi(d\nu).$$

Then we have the following results.

Fact 1: The random sum G in (4) is a stochastic process, $G_t = G_t(\omega)$, $\omega \in \Omega$; i.e. a measurable process [21, p.45].

Fact 2: G is a zero-mean, stationary process.

Fact 3: G is uniformly bounded, $|G_t(\omega)| \le \sqrt{2N}/K_m$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$.

Fact 4: G is asymptotically Gaussian in the limit as $N \to \infty$. In particular, the random variable G_t is, for each t, asymptotically standard Gaussian.

Fact 5: The spectrum of G is $2\pi\Psi$. G has unit variance.

The first three facts are self-evident. Fact 4 follows from the multivariate Central Limit Theorem [22, p.195] while the first part of Fact 5 follows from a calculation similar to one given in [19, p.358]. The variance of G is obtained by integrating the spectrum [23, p.338]:

$$Var[G_t] = \frac{1}{2\pi} \int_0^\infty 2\pi\Psi(d\nu) = 1 \, .$$

The sum (3) is recovered from (4) by the choice $K(\nu) = 1$. Therefore G is a generalization of (3) and is henceforth called Shinozuka noise for easy reference. Notice that the five listed properties of G are independent of the choice of K provided K is Ψ-admissible. K is therefore available for use as a free parameter. Later, after filtering G, we will choose K to fix the ergodicity of the filter output. Five realizations of bandlimited Shinozuka white noise are shown in Fig. 1 together with five realizations of bandlimited Gaussian white noise for comparison. $K(\nu) = \operatorname{sech}\nu$ is used in this example.

Our application of Melnikov theory to random perturbations requires that the noise be continuous uniformly across both time and ensemble. We define a stochastic process X to be ensemble uniformly continuous (EUC) if, given any $\delta_1 > 0$, there exists $\delta_2 > 0$ such that if $t_1, t_2 \in \mathbb{R}$ and $|t_1 - t_2| < \delta_2$ then $|X_{t_1}(\omega) - X_{t_2}(\omega)| < \delta_1$ for all $\omega \in \Omega$. A stochastic process can have UC paths and fail to be EUC. The derivative $G'(\omega)$ of the Shinozuka noise path $G(\omega)$ is bounded,

$$|G'_t(\omega)| \le \frac{1}{K_m} \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \nu_n(\omega)$$

for all $t \in \mathbb{R}$. Thus G is EUC if the sum of its angular frequencies $\{\nu_1, \ldots, \nu_N\}$ is bounded. This sum is bounded if and only if G is bandlimited. Our use of Shinozuka noise requires it to be EUC so we henceforth assume this condition to hold.

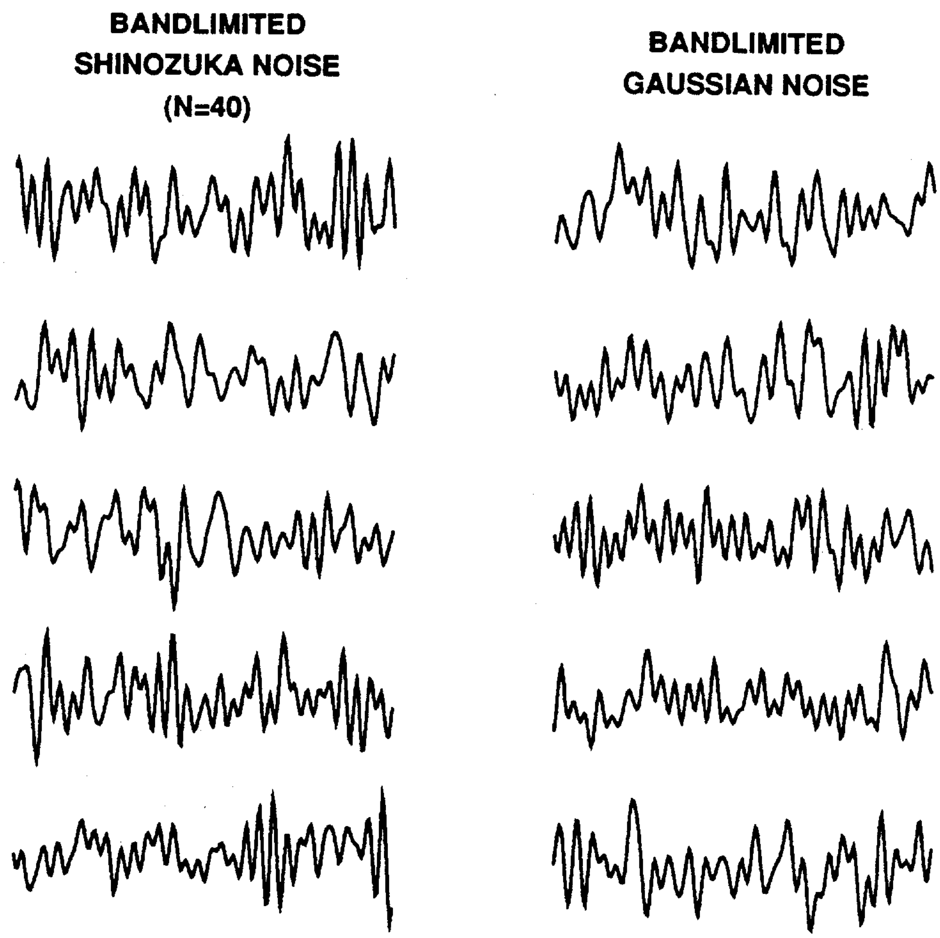

Fig. 1. Realizations of Shinozuka and Gaussian noise processes with identical bandlimited spectra and $K(\nu) = \mathrm{sech}\,\nu$.

3. Melnikov function and phase space flux

The Melnikov function $M(s, \theta_1, ..., \theta_k)$ for bounded, UC excitations $g_1, ..., g_k$ is, according to Meyer and Sell [18], expressed by the Melnikov transform $M(s, \theta_1, ..., \theta_k) = \mathrm{M}[g_1, ..., g_k]$ where s is a reference time and $\theta_1, ..., \theta_k$ are the cross-section times of the Poincaré maps relative to s. The Melnikov function is time-shift invariant and thus can be written $M(s, \theta_1, ..., \theta_k) = M(0, \theta_1 - s, ..., \theta_k - s) \equiv M(t_1, ..., t_k)$. Consider first a fixed noise path $G(\omega)$ given by some particular choice of $\omega \in \Omega$. For our dynamical model with phase space separatrix $\vec{x}_s = (x_s(t), \dot{x}_s(t))$ and bounded, UC excitations g, $G(\omega)$,

$$
\begin{aligned}
M(s, \theta_1, \theta_2; \omega) &= M(t_1, t_2; \omega) \\
&= \mathrm{M}[g, G(\omega)] \\
&= -k \int_{-\infty}^{\infty} \dot{x}_s^2(t)\,dt + \gamma \int_{-\infty}^{\infty} \dot{x}_s(t) g(t+t_1)\,dt + \sigma \int_{-\infty}^{\infty} \dot{x}_s(t) G_{t+t_2}(\omega)\,dt\,.
\end{aligned}
$$

The Melnikov function is related to the distance between the stable and unstable manifolds associated with a hyperbolic fixed point. For such a distance to have meaning the perturbation must be sufficently small that this fixed point persists and remains hyperbolic. This is indeed the case for UC bounded excitations g and $G(\omega)$ and sufficiently small ε [24, Prop. 3.2.2].

Consider now the ensemble of Melnikov functions $M(t_1, t_2) = M(t_1, t_2; \omega)$, $\omega \in \Omega$. Considered sample analytically [16, Appendix A], there exists an ensemble of hyperbolic fixed points corresponding to the ensemble $M(t_1, t_2)$. For $M(t_1, t_2)$ to be meaningful, these fixed points must persist and remain hyperbolic as an ensemble. They do for ε sufficiently small, provided that the ensemble of noise excitations are uniformly bounded and EUC. Bandlimited Shinozuka noise meets these conditions. Thus $M(t_1, t_2)$ meaningfully measures the now random distance between the separated manifolds and has the desirable measurability properties of a random process (random field) in the two time variables t_1, t_2.

Let $h(t) = \dot{x}_s(-t)$. Then

$$
\begin{aligned}
M(t_1, t_2) &= -k \int_{-\infty}^{\infty} \dot{x}_s^2(t)\,dt + \gamma \int_{-\infty}^{\infty} h(t) g(t_1 - t)\,dt + \sigma \int_{-\infty}^{\infty} h(t) G_{t_2 - t}\,dt \\
&= -k \int_{-\infty}^{\infty} \dot{x}_s^2(t)\,dt + \gamma z(t_1) + \sigma Z_{t_2}
\end{aligned}
$$

where $Z = G * h$ and $z = g * h$ are the convolutions of G and g, respectively, with h. Denoting the integral of $\dot{x}_s^2$ by I, we obtain

$$
M(t_1, t_2) = -Ik + \gamma z(t_1) + \sigma Z_{t_2}. \tag{5}
$$

h can be interpreted as the impulse response of a linear time-invariant filter $\mathbb{F}$, here called the orbit filter. Then $Z = G * h$ is the output of $\mathbb{F}$ with input process G and we write $Z = \mathbb{F}[G]$. Likewise, $z = \mathbb{F}[g]$ is the output of the orbit filter with input g. A sufficient condition for $\mathbb{F}$ to be stable - the property that small changes in the input produce only small changes in the output - is that the impulse response be absolutely integrable [25],

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty . \tag{6}$$

Inasmuch as $h(t) = \dot{x}_s(-t)$, this integral is, in the case of a heteroclinic orbit, the distance separating the hyperbolic fixed points of the orbit. For a homoclinic orbit, the integral is twice the distance from the orbit's hyperbolic fixed point to the point on the x-axis of maximum extent of the orbit. We only consider orbits in which these distances are finite hence $\mathbb{F}$ is stable.

Shinozuka noise is stationary and $\mathbb{F}$ is time-invariant hence the process Z is also stationary. In particular, Z is marginally stationary with nontime-varying mean and variance. Shinozuka noise has zero mean so Z has zero mean and

$$E[M(t_1, t_2)] = -Ik + \gamma z(t_1) .$$

The variance σ_Z^2 of Z is calculated using the transfer function H of the orbit filter. We have

$$H(\nu) = \int_{-\infty}^{\infty} h(t) e^{-j\nu t}\, dt \tag{7}$$

and write $S(\nu) = |H(\nu)|$ to denote the modulus of H. S is the imaginary (real) part of H if h is odd (even). The function S plays the same role as that of the relative scaling factors in [9], [10]. Then [23]

$$\sigma_Z^2 = \frac{1}{2\pi} \int_0^{\infty} |H(\nu)|^2 2\pi \Psi(d\nu) = \int_0^{\infty} S^2(\nu)\, \Psi(d\nu) . \tag{8}$$

It follows from (5), then, that the variance of the Melnikov function is

$$Var[M(t_1, t_2)] = \sigma^2 \sigma_Z^2 = \sigma^2 \int_0^{\infty} S^2(\nu)\, \Psi(d\nu) .$$

Having determined the mean and variance of Z, we describe its marginal distribution μ_Z. Consider passing the deterministic signal $\cos(\nu t + \phi)$ through the orbit filter $\mathbb{F}$. The resulting output is $S(\nu)\cos(\nu t + \phi + \theta(\nu))$ where $\theta(\nu)$ is the phase shift caused by filtering. By superposition, the output Z of the orbit filter $\mathbb{F}$ is

$$Z_t = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \frac{S(\nu_n)}{K(\nu_n)} \cos(\nu_n t + \phi_n + \theta(\nu_n)) .$$

Consider choosing $K = S$. The condition (6) is satisfied so S is bounded and $\kappa < \infty$. Ψ has already been assumed to be bandlimited to make G EUC. Now further restrict the spectrum of G so that S is bounded away from zero on the support of Ψ. In many cases including the two considered in Section 7 this restriction is minor; in the case of the second example it is no restriction at all. With this restriction, the choice $K = S$ is Ψ-admissible and we have

$$Z_t = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos(\nu_n t + \phi_n + \theta(\nu_n)) . \tag{9}$$

ϕ_n is uniformly distributed over the interval $[0, 2\pi]$. ϕ_n and $\theta(\nu_n)$ are independent so their sum modulo 2π is also uniformly distributed over $[0, 2\pi]$. Thus the phase shifts $\theta(\nu_n)$, $n = 1, \ldots, N$ can be dropped from (9) without change to the distribution of Z. Hence we write

$$Z_t = \sqrt{\frac{2}{N}} \sum_{n=1}^{N} \cos(\nu_n t + \phi_n).$$

The random variable Z_t is the sum of bounded, independent, identically distributed terms so, by the Central Limit Theorem, its distribution μ_Z has a Gaussian limit ($N \to \infty$) with mean zero and variance σ_Z^2. The spectrum of Z is $2\pi\Psi_o$ where

$$\Psi_o = \frac{1}{\sigma_Z^2} \int_A S^2(\nu)\, \Psi(d\nu).$$

Ψ is assumed to be continuous so Ψ_o is continuous.

The average phase space flux is closely related to the Melnikov function in the case of near-integrability [9], [10]. For small ε, the average phase space flux is $\varepsilon\Phi + O(\varepsilon^2)$ with the flux factor Φ given by

$$\Phi = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} M^+(s, \theta_1, \theta_2)\, ds \tag{10}$$

where the notation $f^+ = \max(f, 0)$ denotes the positive part of the real function f. Substituting (5) into (10) we obtain

$$\Phi = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\sigma Z_{\theta_2 - s} + \gamma z(\theta_1 - s) - Ik]^+\, ds\,. \tag{11}$$

Existence of the limit in (11) depends on the nature of the excitations g and G and of their corresponding convolutions z and Z. This limit exists provided the excitations are asymptotic mean stationary.

4. Asymptotic mean stationarity

A stochastic process $X = X_t,\ t \in I\!R$ is defined to be asymptotic mean stationary (AMS) if the limit

$$\mu_X(A) \equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[1_A(X_t)]dt \tag{12}$$

exists for each real Borel set $A \subset I\!R$. Here 1_A is the indicator function, $1_A(x)=1$ for $x \in A$ and $1_A(x)=0$ otherwise. If the limits in (12) exist then, by the Vitali-Hahn-Saks Theorem [26, p.277], μ_X is a probability measure. μ_X is called the stationary mean (SM) distribution of the process X. The existence of the limits in (12) also implies [27], [28] that for any Borel-measurable function b bounded on the support of μ_X,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[b(X_t)]\, dt = \int_{-\infty}^{\infty} b(x)\mu_X(dx).$$

Asymptotic mean stationarity is a weaker property than ergodicity. If a process X is ergodic [27] then, with probability one,

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} b(X_t)\, dt = \int_{-\infty}^{\infty} b(x)\mu_X(dx). \tag{13}$$

Any stationary process X is AMS. In this case the SM distribution of X is the marginal distribution of the process, $\mu_X(A)=E[1_A(X_t)]=P\{X_t \in A\}$ for any fixed time t. Shinozuka noise G is stationary hence

$$\mu_G(A) = P\{G_0 \in A\} = P\{\sqrt{\frac{2}{N}}\left[\frac{\cos\phi_1}{K(\nu_1)} + \ldots + \frac{\cos\phi_N}{K(\nu_N)}\right] \in A\}.$$

The random variables $\cos\phi_n / K(\nu_n)$, $n = 1, \ldots, N$ are independent and identically distributed with variance 1/2. It follows from the Central Limit Theorem that μ_G is standard Gaussian in the limit as $N \to \infty$.

The definition given above for AMS processes specializes to real functions and, in particular, periodic, quasiperiodic, and almost periodic [29] real functions are AMS. The SM distribution of an AMS function g has the simpler expression

$$\mu_g(A) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1_A(g(t))dt\,.$$

Hence all AMS functions are ergodic. The SM distribution μ_g describes, in a limiting sense, the proportion of time that the function g has values in a given set A. This is illustrated in Fig. 2. The intervals $T_0, T_{\pm1}, T_{\pm2}, \ldots$ are the times in which $g(t) \in A$. The overall proportion of time spent by g in A is

$$\mu_g(A) = \lim_{T\to\infty} \frac{1}{2T} \sum_{k=-\infty}^{\infty} l_T(T_n) \tag{14}$$

where $l_T(T_n)$ is the length of the truncated interval $T_n \cap [-T, T]$. The AMS property ensures that all such limiting proportions exist. This is exactly what is required to guarantee the existence of the limit in (11) and we henceforth assume that g is AMS.

The most commonly considered form of deterministic forcing is periodic forcing. Let us consider some details of this case. Suppose g is periodic with period T. Then g is AMS and its SM distribution (14) simplifies to

$$\mu_g(A) = \frac{1}{T}\int_0^T 1_A(g(t))dt \,. \tag{15}$$

Introducing the random variable U uniformly distributed over the interval $[0, T]$, we have

$$\mu_g(A) = E[1_A(g(U))] = P\{g(U) \in A\} \,. \tag{16}$$

Thus μ_g can be interpreted as the distribution of the random variable $g(U)$. μ_g is, according to (15), the fraction of time g takes values in a given set. The distribution μ_g has a density f readily calculated from (16) in many cases: Suppose g is sinusoidal with period T and initial phase ϕ, i.e., $g(t) = \sin(2\pi t/T + \phi)$. The density f of its SM distribution is then that of the random variable $\sin U$ and we have

$$f(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}, \quad -1 < x < 1 \,. \tag{17}$$

Or suppose g is sawtooth, i.e.,

$$g(t) = \sum_{n=-\infty}^{\infty} \Delta(t - nT + \phi)$$

where $\Delta(t) = 1 - 4t/T$ for $0 \le t < T/2$, $\Delta(t) = 4t/T - 3$ for $T/2 \le t < T$, and $\Delta(t) = 0$, otherwise. In this case, the SM distribution of g has a constant density over the interval $[-1, 1]$. In each case, sinusoidal and sawtooth, the SM distribution is independent of the initial phase ϕ and the period T of g. This is true for all periodic functions.

The assumptions imposed on g and G to this point fix their ergodic properties separately. The joint ergodicity of g and G is addressed by the following lemma (proved in the Appendix.) First we state some definitions. Two stochastic processes X and Y are jointly AMS if the vector process (X, Y) is AMS. Let λ be the SM distribution of the jointly AMS processes X and Y. If X and Y are jointly AMS then X and Y are each separately AMS with respective (marginal) SM distributions $\mu_X(\cdot) = \lambda(\cdot \times \mathbb{R})$ and $\mu_Y(\cdot) = \lambda(\mathbb{R} \times \cdot)$. Two jointly AMS processes X, Y with joint and marginal SM distributions λ, μ_X, μ_Y are said to be AMS independent if the joint distribution λ is the product $\lambda = \mu_X \times \mu_Y$. Finally, X and Y are jointly ergodic if the vector process (X, Y) is ergodic. Then, analogous to (13),

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} b(X_t, Y_t)\, dt = \int_{-\infty}^{\infty} b(x, y)\, \lambda(dx \times dy) \tag{18}$$

with probability one for any Borel-measurable function b bounded on the support of λ.

Lemma 1: I. Let X and Y be independent processes and suppose X is AMS and Y is stationary. Then X and Y are jointly AMS and AMS independent. II. Suppose X and Y are jointly AMS and AMS independent. If X and Y are each ergodic, then (X, Y) is ergodic.

The existence of the limit in (11) for the flux factor Φ is determined from Lemma 1. z is AMS (because g is assumed to be AMS.) Z is stationary so, by Lemma 1, z and Z are jointly AMS. The spectrum of Z is continuous so Z is ergodic. z is also ergodic so z and Z are jointly ergodic, again by Lemma 1. Then, using (18) and identifying b with the integrand in (11), the existence of the limit follows.

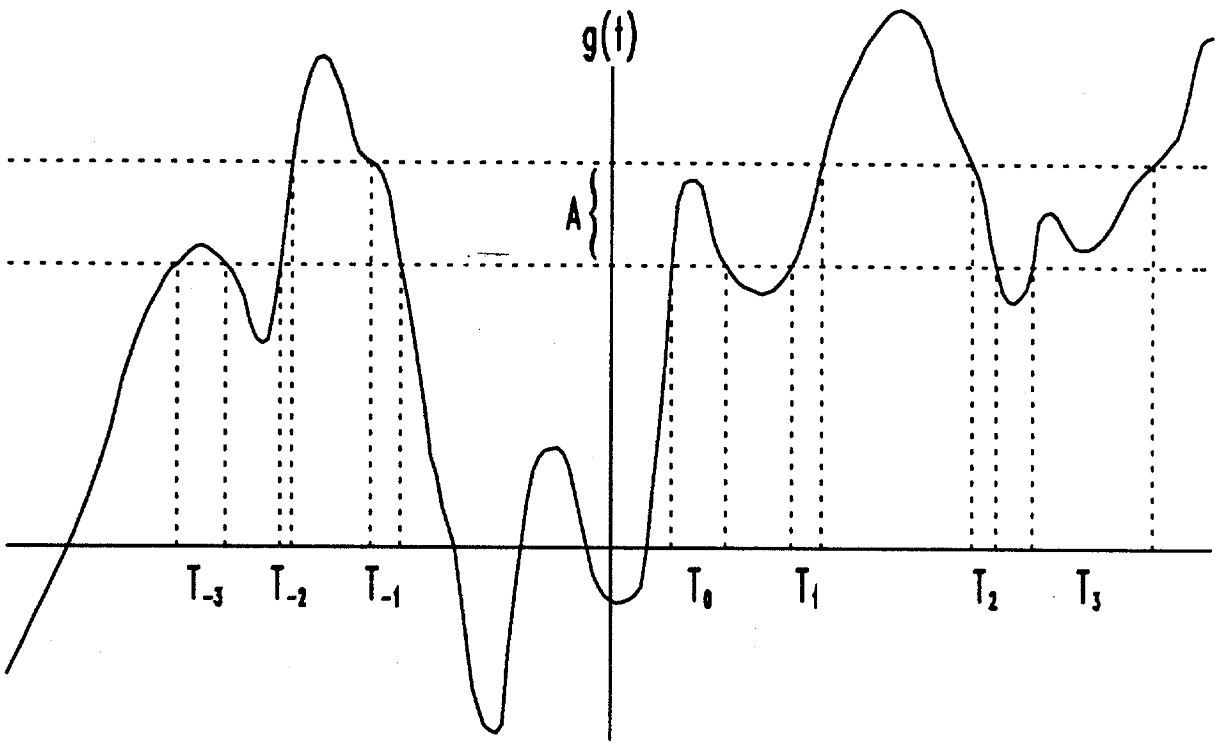

Fig. 2. Intervals $T_{\pm n}$ during which the function g falls in the set A.

5. Calculation of Φ

The random variable Φ appears in (11) to depend on the initial phase parameters θ_1, θ_2. However, this is not so as the following lemma demonstrates.

Lemma 2: Let Z be a stationary stochastic process and suppose that the limit in (11) exists. Then

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\sigma Z_{\theta_2-s}+\gamma z(\theta_1-s)-Ik]^+\,ds \overset{d}{=} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\sigma Z_t+\gamma z(t)-Ik]^+\,dt\,. \tag{19}$$

where $\overset{d}{=}$ denotes equality in distribution.

The proof of Lemma 2 is given in the Appendix. According to Lemma 2, the distribution of Φ depends on the damping and forcing parameters k, γ, and σ, on the forcing function g, and on the spectrum $2\pi\Psi$ of the forcing process G, but is not a function of the relative starting time $\theta_1-\theta_2$ of g and G.

Lemma 2 does not require g to be AMS. If g is AMS then we have the following theorem.

Theorem 1: Suppose g is AMS and Z is stationary and ergodic. Then the limit in (11) exists and the flux factor Φ is nonrandom. In fact,

$$\Phi = E[(\sigma A + \gamma B - Ik)^+] \tag{20}$$

where A is a random variable with distribution equal to the marginal distribution μ_Z of the process $Z=\mathbb{F}[G]$, B is a random variable with distribution equal to the SM distribution μ_z of the process $z=\mathbb{F}[g]$, and A and B are independent.

Proof: g is AMS so [30] the output $z=\mathbb{F}[g]$ of the orbit filter $\mathbb{F}$ is also AMS. Z is stationary so, by Lemma 1, z and Z are jointly AMS and AMS independent. Thus the limit on the r.h.s. of (19) exists. Also by Lemma 1, z and Z are jointly ergodic since they are both separately ergodic. Thus the limit is a constant and the distributional equality in (19) is an equality. The limiting time average in (19) may be replaced, as indicated in (18), by an ensemble average

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\sigma Z_t+\gamma z(t)-Ik]^+\,dt = \int_{\mathbb{R}\times\mathbb{R}}\left(\sigma z_1+\gamma z_2-Ik\right)^+(\mu_z\times\mu_Z)(dz_1\times dz_2)$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\sigma z_1+\gamma z_2-Ik\right)^+\mu_z(dz_1)\mu_Z(dz_2)\,. \tag{21}$$

Equivalently, the ensemble average (21) can be written as an expectation giving, by Lemma 2, the result of the theorem. A and B in (20) are independent as a direct consequence of the product form of the SM distribution $\mu_z\times\mu_Z$ of z and Z.

A striking feature of Formula (20) for Φ is the similarity of the roles played by the deterministic forcing g and stochastic forcing $G - g$ represented by the random variable B and G by the random variable A. The dynamical system cannot distinguish between a deterministic excitation and a stochastic excitation. In the former case it experiences the fixed excitation represented by g; in the latter case it also experiences a particular excitation - that represented by the realization $G(\omega)$ of the process G. Put another way, the dynamical system is unaware of the ensemble of possible excitations represented by the process G; it is aware only of the single excitation in the ensemble to which it is subjected. Since all that matters to the calculation of the flux factor Φ are the SM distributions of the filtered excitations, the roles played by g and G in (20) should be expected to parallel one another.

Let us now evaluate (20). Let f and F be, respectively, the standard Gaussian density function and distribution function

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, \quad F(z) = \int_{-\infty}^{z} f(x)\,dx .$$

Let $X = \sigma A$ and denote its distribution function by F_N, with subscript to remind us that the distribution of X depends on the parameter N of the Shinozuka noise model. Also, let $Y = Ik - \gamma B$. Then,

$$\Phi = E[E[(X-Y)^+ | B]] = E\left[\int_Y^\infty (x-Y)F_N(dx)\right]. \tag{22}$$

For large N, the distribution F_N can be replaced in (22) by the standard Gaussian distribution F with only small error. This is the substance of the following lemma, the proof of which is given in the Appendix.

Lemma 3: As $N \to \infty$,

$$E\left[\int_Y^\infty (x-Y)F_N(dx)\right] \to E\left[\int_Y^\infty (x-Y)F(dx)\right].$$

Assume $\sigma > 0$, so that X is not identically zero. Then, according to Lemma 3,

$$\begin{aligned}
\Phi &\doteq \frac{1}{\sigma\sigma_Z} E\left[\int_Y^\infty (x-Y) f\left(\frac{x}{\sigma\sigma_Z}\right) dx\right] \\
&= \frac{1}{\sigma\sigma_Z} E\left[\int_Y^\infty x f\left(\frac{x}{\sigma\sigma_Z}\right) dx - Y \int_Y^\infty f\left(\frac{x}{\sigma\sigma_Z}\right) dx\right] \\
&= \sigma\sigma_Z E\left[\int_U^\infty u f(u)\,du\right] - E\left[Y \int_U^\infty f(u)\,du\right]
\end{aligned}$$

where $U = Y/(\sigma\sigma_Z)$. Thus, for $\sigma > 0$,

$$\Phi \doteq \sigma\sigma_Z E[f(U) + UF(U) - U]. \tag{23}$$

We have the following result.

Theorem 2: Suppose g is AMS and G is the Shinozuka noise in (4) with parameter N and $\sigma > 0$. Then the flux factor Φ is approximately

$$\Phi \doteq E[(\sigma\sigma_Z \Lambda + \gamma B - Ik)^+] = \sigma\sigma_Z E[f(U) + UF(U) - U]$$

where Λ is a standard Gaussian random variable and $U = \sigma^{-1}\sigma_Z^{-1}(Ik - \gamma B)$. The error in this approximation can be made arbitrarily small by a sufficiently large choice of N.

For the case $\sigma = 0$ where no random excitation is present, (22) simplifies directly to

$$\Phi \equiv \Phi_o = E\left[\int_Y^\infty (x - Y)\delta_0(dx)\right] = E[-Y\,1_{(-\infty,0)}(Y)] = E[(-Y)^+] = E[(\gamma B - Ik)^+]. \tag{24}$$

Here δ_x denotes a distribution with a single atom located at x: $\delta_x(A) = 1_A(x)$.

Consider the case where there is no noise and the deterministic forcing is sinusoidal, $g(t) = \cos(\nu t + \phi)$. In this case, the output $z = g * h$ of the orbit filter $I\!F$ is also sinusoidal with amplitude scaled by $S(\nu) = |H(\nu)|$. Thus the random variable $B' = B/S(\nu)$ has the density (17). Hence, in terms of the dimensionless quantities

$$\Phi_{\text{deter.}} \equiv \frac{\Phi_o}{Ik}, \quad \gamma' \equiv \frac{\gamma S(\nu)}{Ik},$$

we have from (24)

$$\Phi_{\text{deter.}} = E[(\gamma' B' - 1)^+] = \frac{1}{\pi}\int_{1/\gamma'}^{1} \frac{\gamma' x - 1}{\sqrt{1 - x^2}} dx. \tag{25}$$

A plot of $\Phi_{\text{deter.}}$ versus γ' is given in Fig. 3(a).

Now let us turn to the opposite case in which the noise, rather than the deterministic forcing, is the cause of phase space transport. This is the case in which γ is zero or in which the mass of the distribution μ_z is concentrated at zero by the orbit filter. The latter occurs, for instance, when the spectrum of g is located outside the passband of $I\!F$. In this case the distribution of U in (23) is approximately that of the constant $Ik/(\sigma\sigma_Z)$. It follows from (20) that, in terms of the dimensionless quantities

$$\Phi_{\text{stoch.}} \equiv \frac{\Phi}{Ik}, \quad \sigma' \equiv \frac{\sigma\sigma_Z}{Ik},$$

we have

$$\Phi_{\text{stoch.}} \doteq E[(\sigma'\Lambda - 1)^+] = \eta(\sigma') \tag{26}$$

where

$$\eta(\sigma') \equiv \sigma' f(1/\sigma') - 1 + F(1/\sigma').$$

The derivative of $\eta(\sigma')$ is $f(1/\sigma')>0$ so, consistent with intuition, any increase in σ' causes an increase in the scaled flux $\Phi_{stoch.}$. $\Phi_{stoch.}$ is shown in Fig. 3(b) as a function of σ'.

In contrast to $\Phi_{deter.}$, the approximation $\eta(\sigma')$ for $\Phi_{stoch.}$ is positive for all levels of noise. This requires careful interpretation. In fact, because Shinozuka noise has uniformly bounded paths, there is a threshold for $\Phi_{stoch.}$, just as for $\Phi_{deter.}$. Below this threshold, $\Phi_{stoch.}=0$. This threshold is not apparent from the approximation $\eta(\sigma')$ although the plot in Fig. 3(b) suggests such a threshold.

The presence or absence of a threshold notwithstanding, the plots of $\Phi_{deter.}$ and $\Phi_{stoch.}$ are similar - both are asymptotically linear for large forcing and both show little or no flux for small forcing. The difference in the plots is explained by expressions (25) and (26). These expressions differ only in the random variables B' and Λ representing the SM distributions μ_z and μ_Z. Thus the difference in the plots of $\Phi_{deter.}$ and $\Phi_{stoch.}$ is solely due to the difference in μ_z and μ_Z. In particular, the difference in the plots in Fig. 3 does not depend on whether or not the excitation is random. If, instead of the sinusoidal forcing used in $\Phi_{deter.}$, we had considered a deterministic sum of N incommensurable sinusoids, then μ_z, like μ_Z, would be a convolution and approximately Gaussian for large N. Thus the SM distribution of the filtered excitation, not the excitation's deterministic or stochastic nature, determines the average flux.

The approximating expression $\eta(\sigma')$ is derived from the limiting Gaussian distribution F obtained from F_N with $N\to\infty$. Also, Shinozuka noise with N terms and spectrum Ψ is Gaussian in the limit as $N\to\infty$. Thus $\eta(\sigma')$ is arguably the flux factor for a Gaussian excitation with spectrum Ψ. Unfortunately, this cannot be established within the framework of present Melnikov theory since this theory is restricted to excitation processes with uniformly bounded paths. Paths of ergodic Gaussian processes are neither uniformly bounded nor even bounded.

Interpreted as the flux factor for a Gaussian excitation, $\eta(\sigma')$ predicts that even the smallest amount of noise produces a nonzero flux. This is consistent with the observation that even the smallest ergodic Gaussian excitation exhibits arbitrarily large swings and can be expected to eventually drive the system from one basin of attraction to that of a competing attractor - such motion being interpretable as chaotic motion on a single strange attractor.

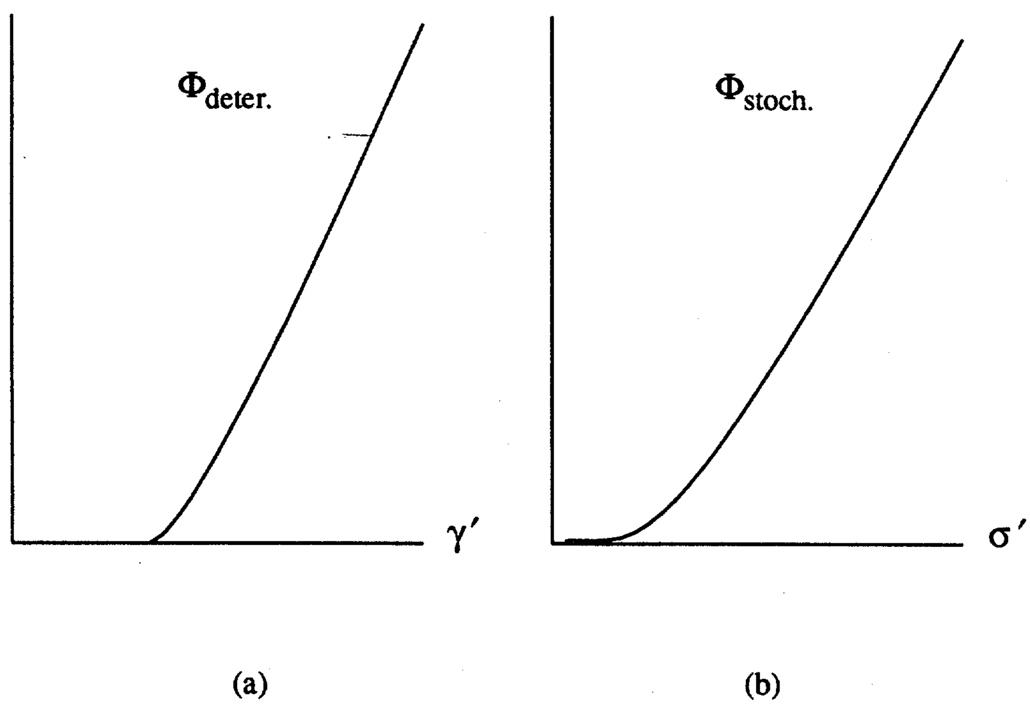

Fig. 3. Dimensionless flux factors. (a) Deterministic forcing. (b) Stochastic forcing.

6. Addition of noise

The addition of a small random excitation into a noiseless, deterministically forced system generally increases the propensity for chaos (as measured by the flux) above that which might otherwise exist. The following lemma determines conditions for $\Phi \geq \Phi_o$. We give the proof in the Appendix.

Lemma 4: Suppose X and Y are independent random variables, $E[Y]=0$, and $\zeta = \zeta(x)$ is convex. Then $E[\zeta(X+Y)] \geq E[\zeta(X)]$ with equality if and only if for almost all x, $\zeta(x)$ is linear over the support of $x+Y$.

The function $\zeta(x)=x^+$ is convex. Thus, comparing (20) and (24), Lemma 4 implies that one always has $\Phi \geq \Phi_o$ for zero-mean, stationary noise. Filtered Shinozuka noise is stationary and zero-mean, thus its presence never reduces the flux factor below that of Φ_o. This directly contradicts a claim of [13] that noise can serve to suppress chaotic behavior which might otherwise be observed in the noise-free system.

The function $\zeta(x)=x^+$ is piece-wise linear so the possibility of equality exists in the conclusion of Lemma 4. Thus, possibly, cases exist in which despite the presence of noise, there is no increase in flux, i.e., $\Phi=\Phi_o$. We now pursue this possibility.

Assume that the SM distribution of the noise has zero mean and connected support. These assumptions are minimal and are satisfied by many noise models including Shinozuka noise. Then, a case-by-case analysis of (20) for all possible distributions of A and B shows that $\Phi=\Phi_o$ in just two cases. The first of these is the case in which $\gamma B + \sigma A \leq Ik$ a.s. In this case, the support of $\gamma B + \sigma A - Ik$ is a subset of $(-\infty, 0]$. $\zeta(x)=x^+$ is linear over this interval and hence satisfies the conditions needed for equality in the conclusion of Lemma 4. Thus we have identified the trivial case in which $\Phi=\Phi_o=0$ - the case in which g and G together fail to produce a nonzero flux.

The other case in which $\Phi=\Phi_o$ is not trivial and requires that the distribution of B have a gap in its support. A simple example of this is the diatomic distribution

$$\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1 \tag{27}$$

composed of two atoms located, as in this case, at 1 and -1 and separated by the gap $(-1,1)$. Suppose that B has the distribution (27) and that $|A| \leq 1$ consistent with the assumption that $|g(t)| \leq 1$. Take $\gamma > Ik + \sigma$. Then $\Phi_o > 0$ because $\gamma > Ik$. If $B=1$, the support of $\sigma A + \gamma B - Ik$ is a subset of $[0, \infty)$. $\zeta(x)=x^+$ is linear over this interval. If $B=-1$, the support of $\sigma A + \gamma B - Ik$ is a subset of $(-\infty, 0]$. $\zeta(x)=x^+$ is linear over this interval, also. Thus, the conclusion of Lemma 4 holds with equality for the present case and we have a nontrivial example in which $\Phi=\Phi_o>0$. Such examples only arise in cases in which a gap exists in the support of B.

Recall that the distribution of B is the SM distribution of the output z of the orbit filter $\mathbb{F}$ with input g. UC AMS functions g with a diatomic SM distribution such as (27) are readily constructed. We give some examples below. But under what conditions does such a distribution persist after filtering by $\mathbb{F}$? This depends on the nature of the orbit filter.

The orbit filter $\mathbb{F}$ is defined by the relation of its impulse response $h(t)=\dot{x}_s(-t)$ to the orbit velocity component $\dot{x}_s$. For one-dimensional systems, $\dot{x}_s$ differs significantly for heteroclinic and homoclinic orbits. In a one-dimensional system, the component $\dot{x}_s$ of a right-hand homoclinic orbit (Fig. 4) increases from zero at $t=-\infty$ and continues positive until, at the maximum point of travel of the orbit, it is again zero. This occurs at time $t=0$. The velocity then swings negative and remains so until the orbit arrives, at $t=\infty$, back at the hyperbolic fixed point from which the orbit originated. If the orbit is left-hand, $\dot{x}_s$ is negative for the first half of the orbit and positive during the return half. In either case - right-hand or left-hand homoclinic orbit - the velocity component is an odd function of time with a single sign change. For a heteroclinic orbit (Fig. 5), $\dot{x}_s$ exhibits no sign change. If the orbit connects one hyperbolic fixed point with another located to the right, then $\dot{x}_s>0$ for all $t \in (-\infty, \infty)$. If the connection is from right to left, $\dot{x}_s<0$ for all $t \in (-\infty, \infty)$. In either case, there is no sign change. This difference - presence or absence of a sign change - has an important consequence for the filter $\mathbb{F}$. Consider the homoclinic case (Fig.4). In this case h is odd and

$$S(0) = \int_{-\infty}^{\infty} h(t)\,dt = 0 .$$

For a heteroclinic orbit (Fig. 5), the above integral is nonzero and $S(0)>0$. The significance of this is that a heteroclinic orbit filter passes the d.c. component of an excitation while a homoclinic orbit filter does not.

Let us consider examples of bounded UC AMS functions g with a diatomic SM distribution such as (27). One trivial example is the function $g(t)=L(t)$ where $L(t)=t$ for $-1 \le t \le 1$, $L(t)=1$ for $t>1$, and $L(t)=-1$ for $t<-1$. Another example is $g(t)=\tanh(t)$. Less trivial examples include functions g which transit between the values -1 and 1 in such a way that, as $t \to \pm\infty$, more time is spent at these two values with less time spent transitting between them. Consider the function $g(t)=L(\sqrt{|t|}\sin\sqrt{t})$. As $t \to \pm\infty$, $g(t)$ remains at ± 1 for longer and longer periods with fewer and fewer transitions between these two values. This function is UC and AMS and also has the diatomic SM distribution (27). The ideas suggested here extend to more than two atoms and to atoms with unequal weights and more generally to the construction of functions g with SM distributions with gapped support. Finally, note also that the SM distribution of the UC AMS function $g(t)=\sin^{1/n}(t)$ is, for large odd integers n, nearly diatomic (Fig. 6).

The preceding examples illustrate the general principle that the existence of atoms in the SM distribution of a function g requires the function to exhibit persistence; i.e., that there be a d.c. component present in the limiting spectrum of g. Remove that d.c. component and the atoms relocate to zero causing the gaps between them to disappear. Thus far we have only explicitly addressed atomic, or discrete, SM distributions. In fact, the same result holds generally for SM distributions with holes. Filter out the d.c. component and all gaps in the SM distribution are lost.

Juxtapose this with the respective effects of homoclinic and heteroclinic orbit filters on the d.c. component of the excitation and it follows that, for the one-dimensional dynamical systems considered here, noise added to deterministic forcing always increases the average flux associated with homoclinic chaos except for trivial cases in which $\Phi_o=0$. In the case of heteroclinic chaos, added noise does not always increase the average flux, i.e. $\Phi=\Phi_o$, even in cases where $\Phi_o>0$. In no case does the noise lower the average flux.

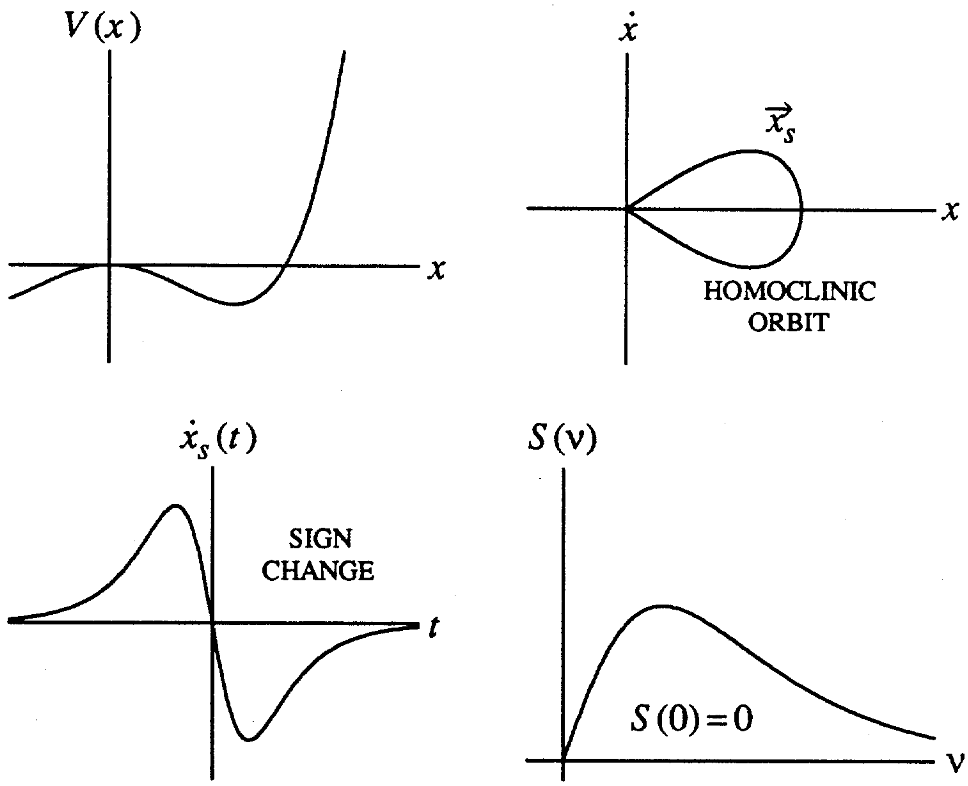

Fig. 4. Filter function $S(\nu)$ for a homoclinic orbit. $S(0)=0$.

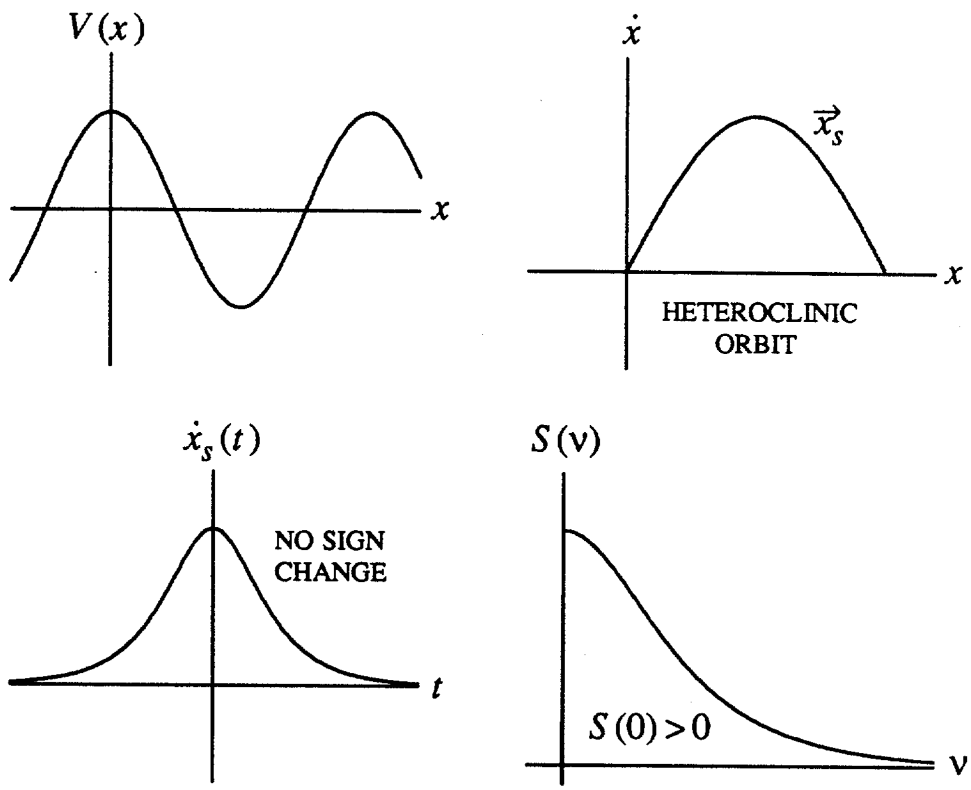

Fig. 5. Filter function $S(\nu)$ for a heteroclinic orbit. $S(0) > 0$.

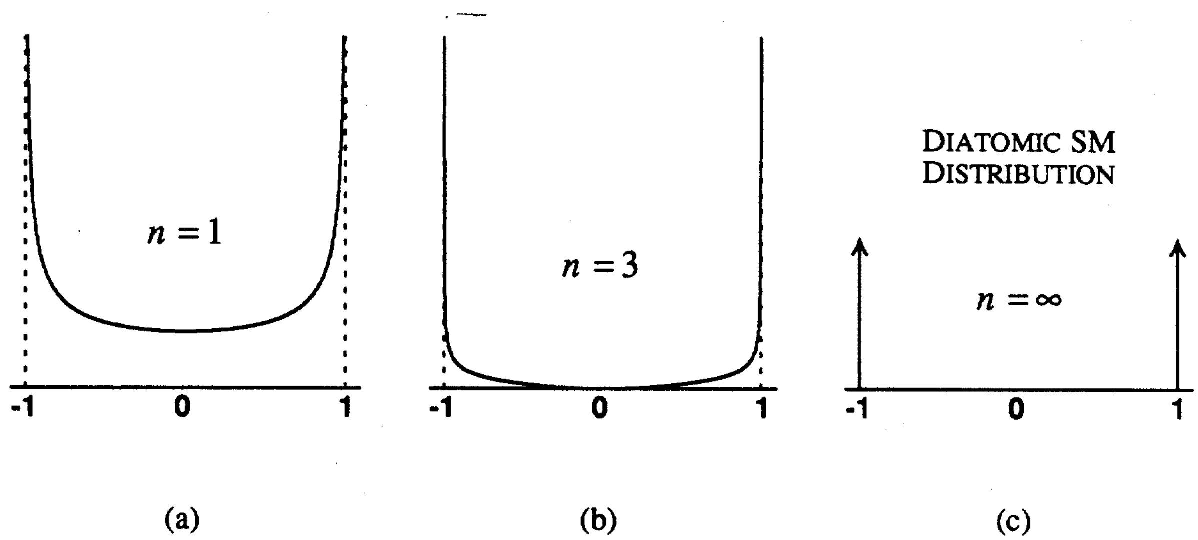

Fig. 6. SM distribution of $g(t) = \sin^{1/n}(t)$.
(a) $n = 1$. (b) $n = 3$. (c) $n = \infty$ (diatomic).

7. Role of the noise spectrum

The spectrum Ψ of the noise affects the flux factor Φ only indirectly through the presence in (20) of the variance

$$\sigma_Z^2 = \int_0^\infty S^2(\nu)\,\Psi(d\nu) \tag{28}$$

of the filtered noise. We have already observed that the average flux depends on the SM distributions of the filtered excitations. These distributions record the proportions of time spent by g and G within given intervals but ignore the order in which the filtered excitations visit different intervals. Autocorrelation is a measure of the order of visitation and is described by the spectrum of the noise. Thus the effect of the proportion of time spent in any given interval by an excitation and the effect of the order in which an excitation visits successive intervals are clearly delineated in Formula (20). The order of visitation, as reflected in the spectrum of the filtered noise, determines the variance of the SM distribution of the filtered excitation. All other features of this distribution are determined by the relative proportion of time spent in any given interval.

Although the role of the noise spectrum Ψ is limited to its effect on the variance, this effect on the flux factor is important. When Ψ is mismatched to the orbit filter $\mathbb{F}$ (i.e., most of the noise energy is outside the passband of the filter), the integral in (28) is close to zero and the flux due to the noise excitation is correspondingly small. Also, because of mediation by the orbit filter, even infinite energy white noise has limited effect on the flux. We illustrate this with two examples: the Duffing-Holmes oscillator for homoclinic chaos and the rf-driven Josephson junction for heteroclinic chaos.

The Duffing-Holmes oscillator [7] is one of the simplest one degree-of-freedom dynamical systems capable of homoclinic chaotic motion and has been extensively studied via mechanical laboratory and numerical computer models as well as analytically. The Duffing-Holmes oscillator is given by (2) with the potential $V(x)=x^4/4-x^2/2$. Thus we have

$$\ddot{x} = x - x^3 + \varepsilon[\gamma g(t) + \sigma G_t - k\dot{x}]\,.$$

The global geometry of the Duffing-Holmes oscillator without damping or external excitation is simple; the unperturbed Duffing-Holmes equation $\ddot{x}=x-x^3$ has a hyperbolic fixed point at the origin $(x,\dot{x})=(0,0)$ in phase space connected to itself by symmetric homoclinic orbits. These orbits are given by

$$\begin{bmatrix} x_s(t) \\ \dot{x}_s(t) \end{bmatrix} = \pm \begin{bmatrix} \sqrt{2}\,\mathrm{sech}\,t \\ -\sqrt{2}\,\mathrm{sech}\,t\,\tanh t \end{bmatrix}.$$

We use the righthand (+) orbit in our flux calculations. The same results obtain for the lefthand orbit.

The impulse response h of the Duffing-Holmes orbit filter is $h(t)=\dot{x}_s(-t)=\sqrt{2}\,\mathrm{sech}\,t\,\tanh t$. Then $I=3/4$. The filter function S for the Duffing-Holmes oscillator is found as in [24] to be $S(\nu)=\sqrt{2}\pi\nu\,\mathrm{sech}(\pi\nu/2)$. Let $\nu_m=\mathrm{argmax}(S)$ with $S_m=S(\nu_m)$. We have $\nu_m\tanh\nu_m=1$ and $S_m^2=2\pi^2\nu_m^2-8=3.514$. Thus $S^2(\nu)\le 3.514$ for all $\nu\ge 0$ and it follows from (28) that, for any noise spectrum, we always have $\sigma_Z \le 1.875$.

An upper bound exists for σ_Z even in the infinite energy case of white Shinozuka noise. Let $\{G^{(l)}, l=1,2,...\}$ be a sequence of independent Shinozuka processes such that the spectrum of $G^{(l)}$ is uniformly distributed over the spectral interval $[l, l+1]$. Form the sum process

$$G_{(W)} = \sum_{l=1}^{W} G^{(l)}.$$

The spectrum $2\pi\Psi^{(W)}$ of the sum $G_{(W)}$ has density $2\pi 1_{[0,W)}(\nu)$. White Shinozuka noise can be considered as the limit of $G_{(W)}$ as $W \to \infty$. For Shinozuka noise with spectrum $2\pi\Psi_{(W)}$ we have

$$\sigma_Z^2 = \int_0^\infty S^2(\nu)\Psi_{(W)}(d\nu) = \int_0^W S^2(\nu)d\nu.$$

Now,

$$\int_0^\infty S^2(\nu)d\nu = \frac{4\pi}{3},$$

so $\sigma_Z \le 2\sqrt{3\pi}/3$ with $\sigma_Z = 2\sqrt{3\pi}/3 = 2.047$ in the white noise limit as $W \to \infty$. Thus, because of the exponential decay of $S(\nu)$ for large ν, even infinite energy white Shinozuka noise only finitely effects the flux factor Φ.

Our second example, the rf-driven Josephson junction [2], [31] is a one degree-of-freedom nonlinear system capable of heteroclinic chaotic dynamics. Its equation of motion is of the form (2) with potential $V(x) = \beta^2\cos x$. The global geometry of the unperturbed system $\ddot{x} = \beta^2\sin x$ features an alternating sequence of elliptic and hyperbolic fixed points regularly spaced π units apart. An expression for the hyperbolic orbit connecting any two neighboring hyperbolic fixed points in this sequence can be given in closed form. By elementary means starting from the Hamiltonian equation $\dot{x}^2/2 + \beta^2\cos x = \beta^2$, the velocity coordinate of the orbit is found to be $\dot{x}_s(t) = 2\beta\operatorname{sech}\beta t$. Thus $h(t) = 2\beta\operatorname{sech}\beta t$, $I = 8\beta$, and $S(\nu) = \pi\operatorname{sech}(\pi\nu/(2\beta))$. $S_m = \pi$ so $\sigma_Z < \pi$ for all possible noise spectra $2\pi\Psi$. Here, as with the Duffing-Holmes oscillator, $S(\nu)$ is bounded and decreases exponentially with ν. Thus, also for the rf-driven Josephson junction, infinite energy white Shinozuka noise has only finite effect on σ_Z and, indirectly, on the flux.

8. Conclusions

The primary achievement of this work is the expression

$$\Phi = E[(\sigma A + \gamma B - Ik)^{+}] \tag{29}$$

given in Theorem 1 for the average flux factor. This expression applies generally to second order, one degree-of-freedom, near-integrable dynamical systems whose unperturbed flow includes homoclinic or heteroclinic orbits. Also, the derivation leading to (29) is readily modified to accommodate third and higher order one degree-of-freedom systems, more general two-dimensional vector field systems, and multiplicative excitation. In all these latter cases the form of (29) remains the same; only the orbit filter $\mathbb{F}$ used to determine the constant I and the distributions of A and B changes.

The framework of the derivation of (29) puts random and deterministic excitations on an equal footing. This is accomplished by requiring that both types of excitation be AMS. This requirement ensures the existence of the average flux and provides a common means - the SM distribution - for studying the two classes of excitation. According to (29), the effect of an excitation on Φ is mediated by the orbit filter with only the SM distribution of the AMS filtered excitation explicitly affecting the average flux. This applies equally to deterministic and random excitations and testifies to a fundamental equivalence of the roles played by the two types of excitation in the promotion of Smale horseshoe-type chaotic motion in our dynamical model.

Expression (29) supports the following conclusions, applicable for sufficiently small ε. According to Lemma 4, the form of (29) indicates that external random excitation never decreases the average phase space transported. In this sense, (29) shows that noise cannot on the average suppress chaotic behavior. Close analysis of (29) and Lemma 4 shows, further, that the presence of noise strictly increases the propensity for homoclinic chaos in all nontrivial cases. For heteroclinic chaos, the same analysis identifies a nontrivial case in which the presence of noise does not change the average flux. In all other nontrivial cases, the propensity for heteroclinic chaos is strictly increased by the presence of noise.

In this work we are constrained by limitations of current Melnikov theory to consider only ensembles of uniformly bounded, EUC excitations. This rules out a direct treatment of Gaussian excitations as no Gaussian process has uniformly bounded paths. Use of the Shinozuka noise model partially circumvents this prohibition. Using (29), the flux factor Φ was calculated for Shinozuka noise and a limit expression was obtained by letting $N \to \infty$. This limit expression was given in Theorem 2 as an approximation for Φ in the case of Shinozuka noise with a large number N of terms. Since Shinozuka noise is Gaussian in the limit as $N \to \infty$, it is reasonable to anticipate that when Melnikov theory is appropriately reformulated to directly address Gaussian excitations, the flux factor for such excitations will be exactly the expression given in Theorem 2.

The approximation for Φ in Theorem 2 concisely identifies the nature and effect of the interaction of the noise spectrum with the orbit filter. The overall impact of the noise on the average flux is determined by how much of the noise energy is located in the filter passband. In fact, the variance σ_Z^2 of the random variable A in (29) representing the filtered noise SM distribution is given by formula (28). In two examples, the Duffing-Holmes oscillator and the rf-driven Josephson junction, (28) was used to show that σ_Z^2 was bounded above - even in the extreme case of infinite energy white noise.

Appendix

Proof of Part 1 of Lemma 1: Suppose X is AMS and Y is stationary and denote their respective SM distributions by μ_X and μ_Y. Also suppose X and Y are independent. It follows that (X,Y) is AMS with SM distribution $\mu_X \times \mu_Y$ provided we show that, for each Borel-measurable subset $A \subset \mathbb{R}^2$,

$$I_T(A) = \frac{1}{2T}\int_{-T}^{T} E[1_A(X_t, Y_t)]dt$$

exists in the limit as $T \to \infty$ with $I_\infty(A) = (\mu_X \times \mu_Y)(A)$. First consider subsets $F \subset \mathbb{R}^2$ which are finite unions of measurable rectangles

$$F = \bigcup_{i=1}^{n} U_i \times V_i .$$

For such F,

$$\begin{aligned}
I_T(F) &= \sum_{i=1}^{n} \frac{1}{2T}\int_{-T}^{T} E[1_{U_i}(X_t)1_{V_i}(Y_t)]\,dt \\
&= \sum_{i=1}^{n} \frac{1}{2T}\int_{-T}^{T} E[1_{U_i}(X_t)]E[1_{V_i}(Y_{t+s})]\,dt \\
&= \lim_{s\to\infty} \frac{1}{2S}\int_{-S}^{S} \sum_{i=1}^{n} \frac{1}{2T}\int_{-T}^{T} E[1_{U_i}(X_t)]E[1_{V_i}(Y_{t+s})]\,dt\,ds \\
&= \sum_{i=1}^{n} \frac{1}{2T}\int_{-T}^{T} E[1_{U_i}(X_t)]\lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S} E[1_{V_i}(Y_{t+s})]\,ds\,dt \\
&= \sum_{i=1}^{n} \mu_Y(V_i)\frac{1}{2T}\int_{-T}^{T} E[1_{U_i}(X_t)]\,dt .
\end{aligned} \tag{A1}$$

The second equality above is due to the independence of X and Y and the stationarity of Y. The fourth equality is obtained by applying Fubini's Theorem [32] to interchange order of integration and then by using the Dominated Convergence Theorem [32] to take the limit inside the integral. Now, letting $T \to \infty$ in (A1), we have

$$I_\infty(F) = \sum_{i=1}^{n} \mu_X(U_i)\mu_Y(V_i) = (\mu_X \times \mu_Y)(F). \tag{A2}$$

The Approximation Theorem [32] is now used to show that (A2) holds for all Borel subsets of $\mathbb{R}^2$. Let A be a fixed Borel subset. By the Approximation Theorem, there exist finite unions F_1, F_2 of measurable rectangles such that, for any $\varepsilon > 0$, $F_1 \subset A \subset F_2$ with $(\mu_X \times \mu_Y)(F_2) \le (\mu_X \times \mu_Y)(A) + \varepsilon$ and $(\mu_X \times \mu_Y)(A) \le (\mu_X \times \mu_Y)(F_1) + \varepsilon$. We use this construction together with (A2) to show that the difference between $I_\infty(A)$ and $(\mu_X \times \mu_Y)(A)$ is arbitrarily small. Now

$$|I_\infty(A) - (\mu_X \times \mu_Y)(A)| \le |I_\infty(A) - I_\infty(F_1)| + |I_\infty(F_1) - (\mu_X \times \mu_Y)(A)| . \tag{A3}$$

Regarding the first term on the r.h.s. of (A3), we have

$$|I_\infty(A) - I_\infty(F_1)| \le I_\infty(F_2) - I_\infty(F_1) = (\mu_X \times \mu_Y)(F_2) - (\mu_X \times \mu_Y)(F_1) < 2\varepsilon .$$

The second term on the r.h.s. of (A3) is equal to $(\mu_X \times \mu_Y)(A) - (\mu_X \times \mu_Y)(F_1)$ and $(\mu_X \times \mu_Y)(A) - (\mu_X \times \mu_Y)(F_1) < \varepsilon$. This proves the first part of the lemma.

Proof of Part II of Lemma 1: Suppose that X and Y are jointly AMS and AMS independent with marginal and joint SM distributions μ_X, μ_Y, and $\mu_X \times \mu_Y$. Suppose, in addition, that X and Y are each ergodic. Let m_X, m_Y, and m_{XY} be the marginal and joint measures of the processes X and Y and let $\mathbb{B}_X$, $\mathbb{B}_Y$, and $\mathbb{B}_{XY}$ be the σ-fields of invariant sets [33] of X, Y, and (X, Y) respectively. A process is ergodic if and only if each of its invariant sets has measure zero or one [33]. For example, X is ergodic if and only if $m_X(E) = 0, 1$ for all $E \in \mathbb{B}_X$. Consider $\mu_{XY}(G)$ where $G \in \mathbb{B}_{XY}$ is a union of disjoint $\mathbb{B}_X \times \mathbb{B}_Y$-measurable rectangles

$$G = \bigcup_{i=1}^{n} E_i \times F_i . \tag{A4}$$

X and Y are jointly AMS and AMS independent so [27, Lemma 6.3.1]

$$m_{XY}(G) = (\mu_X \times \mu_Y)(G) = \sum_{i=1}^{n} \mu_X(E_i)\mu_Y(F_i) .$$

X is AMS so $\mu_X(E_i) = m_X(E_i)$ for each $i = 1, \ldots, n$. Similarly, $\mu_Y(F_i) = m_Y(F_i)$ for each $i = 1, \ldots, n$. Therefore

$$m_{XY}(G) = \sum_{i=1}^{n} m_X(E_i) m_Y(F_i) .$$

X is ergodic so $m_X(E_i) = 0, 1$ for each $i = 1, \ldots, n$. Likewise, $m_Y(F_i) = 0, 1$ for each $i = 1, \ldots, n$. Therefore, $\mu_{XY}(G) = 0, 1$. The sets G of the form in (A4) generate $\mathbb{B}_{XY}$ consequently $\mu_{XY}(G) = 0, 1$ for all $G \in \mathbb{B}_{XY}$. Therefore (X, Y) is ergodic.

Proof of Lemma 2: Let L denote the limit on the l.h.s. of (19). We have

$$\begin{aligned} L &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T-\theta_1}^{T-\theta_1} [\sigma Z_{\theta_2-\theta_1-t} + \gamma z(-t) - Ik]^+ \, dt \\ &= \lim_{T\to\infty} \frac{T+\theta_1}{2T} \frac{1}{T+\theta_1} \int_{-T-\theta_1}^{0} [\sigma Z_{\theta_2-\theta_1-t} + \gamma z(-t) - Ik]^+ \, dt \\ &\qquad + \lim_{T\to\infty} \frac{T-\theta_1}{2T} \frac{1}{T-\theta_1} \int_{0}^{T-\theta_1} [\sigma Z_{\theta_2-\theta_1-t} + Ik - \gamma z(-t)]^+ \, dt . \end{aligned} \tag{A5}$$

Now

$$\frac{T+\theta_1}{T} \to 1 , \quad \frac{T-\theta_1}{T} \to 1$$

as $T \to \infty$ so, substituting T for $T+\theta_1$ in the first integral in (A5) and T for $T-\theta_1$ in the second integral, we obtain

$$L = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\sigma Z_{\theta_2-\theta_1-t} + \gamma z(-t) - Ik]^+ \, dt \, .$$

Then by a change of variable,

$$L = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\sigma Z_{\theta_2-\theta_1+t} + \gamma z(t) - Ik]^+ \, dt \, .$$

The process Z is stationary so

$$L \stackrel{d}{=} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\sigma Z_t + \gamma z(t) - Ik]^+ \, dt \, .$$

Proof of Lemma 3: Define $Z_N = (X_N - Y)^+$ and $Z = (X - Y)^+$ where $X_N = \sigma A$, $Y = Ik - \gamma B$, and X is standard Gaussian. We need to show that

$$E[Z_N] \to E[Z] \tag{A6}$$

as $N \to \infty$. By the Central Limit Theorem and the independence of X_N and Y, we have $(X_N, Y) \xrightarrow{d} (X, Y)$. Thus, also, $Z_N \xrightarrow{d} Z$. Uniform integrability of $\{Z_N\}$ then establishes (A6). A sufficient condition for uniform integrability is [34] that

$$\sup_N E[Z_N^2] < \infty . \tag{A7}$$

In the present case, $E[Z_N^2] \le E[(X_N - Y)^2] = \sigma^2\sigma_Z^2 + E[Y^2] < \infty$. Therefore $\{Z_N\}$ is uniformly integrable and (A7) is true.

Proof of Lemma 4: We use Jensen's inequality [32]:

$$\begin{aligned} E[\zeta(X+Y)] &= E[E[\zeta(X+Y)|X]] \\ &\ge E[\zeta(E[X+Y|X])] \\ &= E[\zeta(X+E[Y])] \\ &= E[\zeta(X)] . \end{aligned}$$

The condition for equality follows directly from Jensen's inequality.

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We study the effect of additive noise on near-integrable second-order dynamical systems whose unperturbed flows have homoclinic or heteroclinic orbits. The noise is represented by a type of Shinozuka stochastic process capable of arbitrarily closely approximating Gaussian noise with any specified spectrum. We derive a formula for the flux factor applicable for any asymptotic mean stationary excitation. This derivation shows that, to first order, the effect of the external excitation on the system is mediated by a linear filter associated with the system homoclinic or heteroclinic orbit. It also shows that the stationary mean distribution of the filtered excitation determines the average phase space flux. This is true for both random and nonrandom excitations and indicates that, for the dynamical systems considered here, these two classes of excitation play substantially equivalent roles in the promotion of chaos.

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